CLIFFORD HOPF GEBRA FOR TWO-DIMENSIONAL SPACE*

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Abstract

A Clifford algebra $\mathcal{C}\ell(V, \eta \in V^* \otimes V^*)$ jointly with a Clifford cogebra $\mathcal{C}\ell(V, \xi \in V \otimes V)$ is said to be a Clifford biconvolution $\mathcal{C}\ell(\eta, \xi)$. We show that a Clifford biconvolution for $\dim_{\mathbb{R}} \mathcal{C}\ell = 4$ does possess an antipode iff $\det(\mathrm{id} - \xi \circ \eta) \neq 0$. An antipodal Clifford biconvolution is said to be a Clifford Hopf gebra.

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Contents

1 Introduction

After Bourbaki [1989 §11] we use *cogebra, bigebra* and Hopf *gebra* instead of coalgebra, bialgebra and Hopf algebra.

Let C be an $\mathbb{R}$-space and C^* be an $\mathbb{R}$-dual $\mathbb{R}$-space. If C is an $\mathbb{R}$-cogebra and A is an $\mathbb{R}$-algebra then the $\mathbb{R}$-space $A \otimes C^*$ inherits the structure of an $\mathbb{R}$-algebra with a convolution product: this is a convolution $\mathbb{R}$-algebra, an $\mathbb{R}$-convolution for short. A dual $\mathbb{R}$-space $C \otimes A^*$ inherits a structure of an $\mathbb{R}$-cogebra with coconvolution coproduct: this is a coconvolutional $\mathbb{R}$-cogebra, an $\mathbb{R}$-coconvolution for short.

In particular if an $\mathbb{R}$-space V carries an $\mathbb{R}$-biconvolution algebra & cogebra structure, then do also the $\mathbb{R}$-spaces $\operatorname{End} V \simeq V \otimes V^*$, $\operatorname{End} V^*$, as well as all iterated $\mathbb{R}$-spaces $\operatorname{End} \ldots \operatorname{End} V$ inherit also $\mathbb{R}$-biconvolution algebra & cogebra structures.

If the $\mathbb{R}$-space C is an $\mathbb{R}$-cogebra with a coproduct $\triangle : C \to C \otimes C$, then C^* is an $\mathbb{R}$-algebra with product $\triangle^* : C^* \otimes C^* \to C^*$. If an $\mathbb{R}$-space A is a finite dimensional $\mathbb{R}$-algebra having a binary product $m : A \otimes A \to A$, then an $\mathbb{R}$-dual $\mathbb{R}$-space A^* (or $\mathbb{Z}$-graded dual in the case A is not a finite dimensional $\mathbb{R}$-space) is an $\mathbb{R}$-cogebra with a binary coproduct $m^* : A^* \to A^* \otimes A^*$.

However there are several important situations (free tensor algebra, exterior algebra, Clifford algebra, Weyl algebra, ...) where the dual space of an algebra is also an algebra in a natural way by construction. If this is the case, then by the above ($\mathbb{Z}$-graded) duality, both mutually dual $\mathbb{R}$-spaces carry both structures, algebra & cogebra, and therefore we have a dual pair of $\mathbb{R}$-biconvolutions.

An unital and associative convolution possessing an (unique) antipode is said to be a Hopf gebra or an *antipodal convolution* (Definition 2.3). This terminology has been introduced in [Oziewicz 1997, 2001; Cruz & Oziewicz 2000] and is different from Sweedler's [1969 p. 71]. A general theory of the finite-dimensional antipodal and antipode-less biconvolutions (convolutions and coconvolutions) has been initiated in [Cruz & Oziewicz 2000].

Nill in 1994 and Böhm & Szlachányi in 1996 introduced weak bigebras and weak Hopf gebras with antipode S defined as the Galois connection with respect to the binary convolution $*$ which does not necessarily needs to be unital [Nill 1998, Nill et all. 1998],

$$\mathrm{id} * S * \mathrm{id} = \mathrm{id}, \quad S * \mathrm{id} * S = S.$$

If $\eta \in V^{*\otimes 2}$ is invertible then $\eta^{-1} \in V^{\otimes 2}$. If $\mathcal{C}\ell(V^*, \xi \in V^{\otimes 2})$ is a Clifford $\mathbb{R}$-algebra, then a dual $\mathbb{R}$-space $\mathcal{C}\ell(V, \xi) \equiv \{\mathcal{C}\ell(V^*, \xi)\}^*$ is a Clifford $\mathbb{R}$-cogebra. It was shown in [Oziewicz 1997, and ff.] that a Clifford convolution $\mathcal{C}\ell(\eta, \eta^{-1}) \equiv \mathcal{C}\ell(V, \eta \in V^{*\otimes 2}, \eta^{-1} \in V^{\otimes 2})$ is antipode-less.

The aim of this paper is to show that a Clifford convolution $\mathcal{C}\ell(\eta, \xi) \equiv \mathcal{C}\ell(V, \eta, \xi)$ for $\dim_{\mathbb{R}} V = 2$, *i.e.* for $\dim_{\mathbb{R}}(V^\wedge) = \dim_{\mathbb{R}} \mathcal{C}\ell(V) = 4$, does posses an antipode iff $\det(\mathrm{id} - \xi \circ \eta) \neq 0$ (Main Theorem 4.1). Applications to physics have been proposed in [Fauser 2000b].

2 Biconvolution and antipode

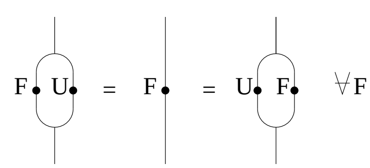

Figure 1: Unital convolution. We assume $U = u \circ \epsilon \in \mathrm{End}\, V$.

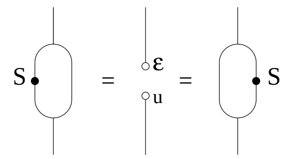

Figure 2: Axioms for the antipode S.

Let the convolution be unital, Fig. 1. This is the case if an $\mathbb{R}$-algebra A is unital with unit $u : \mathbb{R} \to A$ and an $\mathbb{R}$-cogebra C is counital with counit $\epsilon : C \to \mathbb{R}$.

Definition 2.1 (Sweedler 1969 p. 71, Zakrzewski 1990 p. 357). The convolutive inverse, w.r.t. the convolutive unit $U = u \circ \epsilon \in \operatorname{End} V$, of the identity map on V, Fig. 2, is said to be an *antipode*, $S \equiv (\mathrm{id})^{-1}$.

If an antipode exists w.r.t. an unital associative convolution it must be unique.

An antipodal biconvolution defines a unique crossing as given in Fig. 8 in the last Section, see [Oziewicz 1997, 2001; Cruz & Oziewicz 2000]. Using the axioms of the antipode, Fig. 2, and biassociativity, this crossing is equivalent to the algebra homomorphism between algebra and crossed algebra, as well as, to the cogebra homomorphism from crossed cogebra to cogebra. A proof is given in [Cruz & Oziewicz 2000].

Example 2.2 (Graßmann Hopf gebra). The Graßmann wedge product, the Graßmann coproduct and the unique antipode, as given by Sweedler [1969, Ch. XII] and extended by Woronowicz [1989, §3], see Section 3.1 below for details, gives the Graßmann Hopf gebra.

This motivates the following definition:

Definition 2.3. An unital and associative convolution possessing a (unique) antipode is said to be a Hopf gebra or an antipodal convolution.

A closed structure is given by the evaluation and coevaluation [Kelly & Laplaza 1980; Lyubashenko 1995]:

$$V^* \otimes V \xrightarrow{\mathrm{ev}} \mathbb{R},$$
$$V^* \otimes V \xleftarrow[\mathrm{coev}]{} \mathbb{R}.$$

Following [Lyubashenko 1995, fig. 3, p. 250] the evaluation is represented by cup and coevaluation by cap. The evaluation intertwines the transposed endomorphisms $F^* \in \operatorname{End} V^*$ with $F \in \operatorname{End} V$, Fig. 3. Up and down arrows indicate the spaces V^* and V. A coconvolution is counital if e.g. there exits

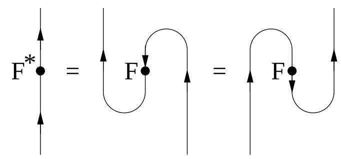

Figure 3: Transposition [Lyubashenko 1995, fig 1 on p. 249].

a dual counit $u^* : V^{*\wedge} \mapsto \mathbb{R}^*$ and a dual unit $\epsilon^* : \mathbb{R}^* \mapsto V^{*\wedge}$, Fig. 4. We

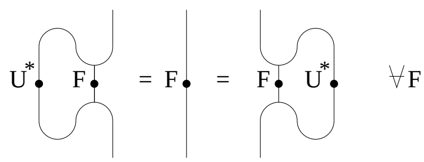

Figure 4: Coconvolutive counit $U^* \in \operatorname{End} V^{*\wedge}$.

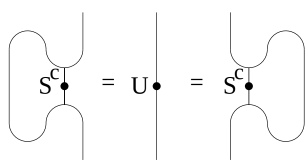

Figure 5: Coconvolutive coantipode S^c.

have to use the dual coconvolution counit $U^* \in \operatorname{End} V^{*\wedge}$ in Fig. 4.

3 Rota & Stein's cliffordization

3.1 Exterior Graßmann Hopf gebra

The Graßmann Hopf gebra was constructed by Sweedler [1969, Ch. XII] factoring the couniversal shuffle tensor Hopf gebra $\mathrm{Sh}V$ by the *switch*, $s(x \otimes y) \equiv y \otimes x$. This construction was generalized to any braid by Woronowicz

[1989, §3, p. 154]. An exterior Hopf gebra can be defined in terms of an unique braid dependent homomorphism of universal tensor Hopf gebra into couniversal tensor Hopf gebra and this implies that an exterior Hopf gebra is couniversal and braided [Oziewicz, Paal & Różański 1995 §8; Różański 1996; Oziewicz 1997 p. 1272-1273].

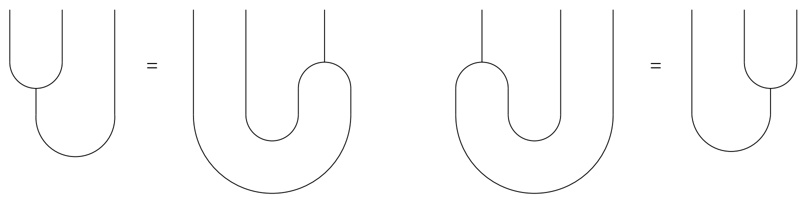

Figure 6: Left-right product - coproduct duality. Cup's are either 'ev' or $\eta^\wedge$.

3.2 Cliffordization

The tensors $\eta, \eta^T \in V^* \otimes V^*$ are said to be scalar products on V or coscalar products on V^* (η^T is the transpose of η). The tensors $\xi, \xi^T \in V \otimes V$ are said to be scalar products on V^* and coscalar on V. In particular $\frac{1}{2}(\eta + \eta^T)$ is the symmetric part of η. The scalar products are displayed by decorated (or labelled) cups and coscalar products by decorated caps, see Fig. 7.

Rota and Stein [1994] introduced the Clifford product as a deformation of exterior biconvolution. This deformation process was called *cliffordization*. A cliffordization introduces an internal loop in a binary product having two inputs and one output, employing the $\eta^\wedge$-cup (or $\xi^\wedge$-cap for the coproduct), Fig. 7.

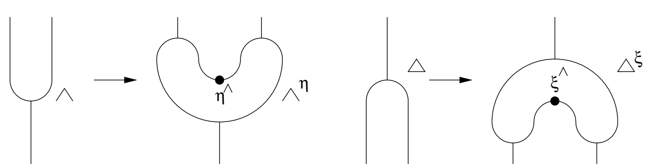

Figure 7: Bicliffordization: the sausage graphs.

Clifford biconvolution was defined in [Oziewicz 2001] as the (η, ξ)-bicliffordization

of an exterior biconvolution. In Sweedler's notation,

$$\wedge^{\eta}(x \otimes y) := x_{(1)} \wedge \eta^{\wedge}(x_{(2)} \otimes y_{(1)}) \wedge y_{(2)},$$
$$\Delta^{\xi} 1 = \xi^{\wedge} \quad \text{and} \quad \triangle^{\xi} x := x_{(1)} \wedge \xi^{\wedge} \wedge x_{(2)}.$$

If the product coproduct duality of Fig. 6 is used with cup as an evaluation, then every product on $V^{\wedge}$ induces a coproduct on $V^{*\wedge}$ and vice versa. If $\eta^{\wedge}$-cup's and ξ-cap's are used, one gets a correlation between products on $V^{\wedge}$ and coproducts on $V^{\wedge}$.

A Clifford $\mathbb{R}$-algebra together with a Clifford $\mathbb{R}$-cogebra on $V^{\wedge}$, $\mathcal{C}\ell(\eta,\xi) \equiv \mathcal{C}\ell(V, \wedge^{\eta}, \Delta^{\xi})$, is said to be a Clifford $\mathbb{R}$-convolution. It was shown in [Oziewicz 1997] that a Clifford $\mathbb{R}$-convolution for $\xi \circ \eta = \mathrm{id}$ and for $\eta \circ \xi = \mathrm{id}$ is antipode-less. An antipode-less Clifford convolution $\mathcal{C}\ell(\eta, \eta^{-1})$ for an *invertible* tensor η *cannot* be a deformation of an exterior Graßmann Hopf gebra.

Definition 3.1. The tensors η and ξ are said to be dependent if $0 \neq A \in \mathrm{End}\, V$ and $0 \neq B \in \mathrm{End}\, V^{*}$ exist such that one of the following relations hold,

$$\xi \;=\; A \circ \eta^{-1} \circ B, \qquad \eta \;=\; B \circ \xi^{-1} \circ A.$$

If the tensors $\eta \,\&\, \xi$ are independent then the Clifford product and coproduct are defined independently by Rota & Stein's *deformation*.

Cliffordization and cocliffordization of the Graßmann convolution does not change the convolutive unit $U = u \circ \epsilon$. However since the $\mathbb{Z}$-grading is changed due to the skewsymmetric parts of η and ξ, the counit is no longer the projection onto $\mathbb{R} \subset V^{\dot{\wedge}}$ [Fauser 1998-2000a].

4 The Clifford antipode

Main Theorem 4.1. A Clifford biconvolution $\mathcal{C}\ell(\eta,\xi)$ is a Clifford Hopf gebra iff $\mu \equiv \det(\mathrm{id} - \xi \circ \eta) \neq 0$. Then

$$
\begin{aligned}
&(i) && \mu S|_V = -\mathrm{id}_V.\\
&(ii) && \det(\mu S) = (-1)^{\dim V}.\\
&(iii) && \mathrm{tr}(\mu S) = \tfrac{1}{2}\mathrm{tr}\{(\eta - \eta^T) \circ (\xi - \xi^T)\}.\\
&(iv) && \text{The minimal polynomial of } \mu S \text{ is}\\
& && (\mu S + 1)[(\mu S - 1)^{\dim V} - \mathrm{tr}(\mu S) \cdot \mu S].
\end{aligned}
$$

Proof. Theorem 4.1 was proved for $\dim_{\mathbb{R}} V = 1$ in [Oziewicz 1997]. If $\dim_{\mathbb{R}} V = n$ then for $A \in \operatorname{End} V$,

$$\det(\lambda \cdot \mathrm{id} - A) = \det(-A) + \ldots + \lambda^{n-2}\tfrac{1}{2}\left\{(\mathrm{tr}A)^2 - \mathrm{tr}(A^2)\right\} - \lambda^{n-1}\mathrm{tr}A + \lambda^n.$$

Therefore $\det(\mathrm{id} - \eta \circ \xi) = \det(\mathrm{id} - \xi \circ \eta)$.

The Clifford antipode $S \equiv S(\eta, \xi) \in \operatorname{End} V^\wedge$ is computed from Fig. 2. We give the proof for $\dim_{\mathbb{R}} V = 2$ only. The general case will be treated elsewhere. Let $r, s, t, u, v, z \in \mathbb{R}$ and

$$\begin{aligned} \eta e_1 &= +r^2\epsilon^1 + t\epsilon^2, & \xi\epsilon^1 &= +u^2 e_1 + z e_2, \\ \eta e_2 &= -t\epsilon^1 - s^2\epsilon^2, & \xi\epsilon^2 &= -z e_1 - v^2 e_2. \end{aligned} \tag{1}$$

Then we find together with (i) of the main theorem:

$$\begin{aligned} \mu \cdot S\,1 \quad &= 1 + 4zt + 2t\, e_1 \wedge e_2, \\ \mu S(e_1 \wedge e_2) &= \quad 2z \quad + \quad e_1 \wedge e_2. \end{aligned}$$

An action of $g \in GL(V)$ on tensors $\eta \in V^* \otimes V^*$ and $\xi \in V \otimes V$ is given as follows

$$\eta \mapsto g^T \circ \eta \circ g, \qquad \xi \mapsto g \circ \xi \circ g^T.$$

It would be desirable to study $GL(V)$-orbits on $(V^* \otimes V^*) \times (V \otimes V)$ and present full classification of all orbits in terms of invariants. However, this topic exceeds the scope of this paper and will be presented elsewhere. □

Remark 4.2. Theorem 4.1 solves and improves Conjecture 2.2 posed in [Oziewicz 1997, p. 1270].

Remark 4.3. The above Clifford Hopf gebra includes as a particular case for $\xi = 0$ the construction made by Đurđevich [1994].

Remark 4.4. In the case $\det \eta \neq 0$ one can take $\xi = \eta^{-1}$. Also if $\det \xi \neq 0$ one can choose $\eta = \xi^{-1}$. In order to prevent such possibilities we need to supplement the definition of the Clifford Hopf gebra with the extra conditions:

$$\det(\mathrm{id} - \xi \circ \eta) \neq 0, \quad \det \xi = \det \eta = 0.$$

The antisymmetric tensor $F \equiv \frac{1}{2}(\eta - \eta^T)$ can be adjusted in such a way that $\det \eta = 0$ with invertible symmetric tensor $g \equiv \frac{1}{2}(\eta + \eta^T)$. We found a Clifford antipode for $\det \eta = 0 = \det \xi$,

$$\begin{gathered} \det(\mathrm{id}_V - \xi \circ \eta) = -\mathrm{tr}(\xi \circ \eta) + 1, \\ t = \pm rs, \quad z = \pm uv; \qquad \mathrm{tr}(\xi \circ \eta) = (ur \pm sv)^2 \neq 1. \end{gathered}$$

The following matrices for $rs \neq 0$ represent the same tensor from $V^{*\otimes 2}$ or from $V^{\otimes 2}$ with respect to the different bases, these matrices are on the same $GL(2, \mathbb{R})$ orbit,

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \simeq \begin{pmatrix} r^2 & rs \\ -rs & -s^2 \end{pmatrix} \simeq \begin{pmatrix} r^2 & 2r^2 s \\ 0 & 0 \end{pmatrix}.$$

An antipode for regular scalar and coscalar tensors can be found also.

5 An antipode-less Clifford bigebra

A Clifford biconvolution $\mathcal{C}\ell(\eta, \xi)$ is antipode-less if $\det(\mathrm{id} - \xi \circ \eta) = 0$. In particular this is the case if $\xi = \eta^{-1}$. It was shown [Oziewicz 1997] that $\wedge^\eta \circ \Delta^{\eta^{-1}} = (\dim \mathcal{C}\ell) \cdot \mathrm{id}_{\mathcal{C}\ell}$.

Problem 5.1. What axioms for Clifford biconvolution implies the condition $\det(\mathrm{id} - \xi \circ \eta) = 0$? In particular, does a braid exists for which such a Clifford biconvolution is a braided Hopf gebra in the usual sense? If such a braid exits how much freedom remains for choices? Compare with [Oziewicz 1997, p. 1272] where it was shown that for $\dim_{\mathbb{R}} V = 1$, $\dim_{\mathbb{R}} \mathcal{C}\ell = 2$ exists a 12-parameter family of crossings.

Lemma 5.2. Let $A \in \mathrm{End}_{\mathbb{R}} V$ and $\dim_{\mathbb{R}} V = 2$. Then the following equations are equivalent:

$$\begin{aligned} (i) \quad & \det(\mathrm{id}_V - A) \equiv \det A - \mathrm{tr} A + 1 = 0, \\ (ii) \quad & (\mathrm{id}_V - A) \circ (\mathrm{id}_V + A - \mathrm{tr} A) = 0. \end{aligned}$$

Proof. $A^2 = (\mathrm{tr}\, A)A - (\det A)\mathrm{id}_V$. □

According to Lemma 5.2 we have to ask that

$$\begin{array}{rl} \text{either} & \mathrm{im}(\mathrm{id}-A) \subset \ker(\mathrm{id}+A-\mathrm{tr}A) \\ \text{or} & \mathrm{im}(\mathrm{id}+A-\mathrm{tr}A) \subset \ker(\mathrm{id}-A). \end{array}$$

We present three examples of antipode-less Clifford biconvolutions $\mathcal{C}\ell(\eta,\xi)$, $\det(\mathrm{id}-\xi\circ\eta)=0$ for $\dim_{\mathbb{R}} V=2$, $\dim_{\mathbb{R}} V^{\wedge}=4$ and for $\eta\,\&\,\xi$ given by (1) with signature $(+,-)$,

$$tr(\mu S)=\tfrac{1}{2}\mathrm{tr}\{(\eta-\eta^T)\circ(\xi-\xi^T)\}=4tz,$$

$$\begin{array}{rl} \text{Case I.} & \eta^T=\eta, \quad \xi^T=\xi \text{ and } \det(\eta\circ\xi)+1=\mathrm{tr}(\eta\circ\xi). \\ \text{Case II.} & r^2=0, \quad \mathrm{tr}(\xi\circ\eta)+1+tz=0, \quad z+t\det\xi=0. \\ \text{Case III.} & v^2=0, \quad tz=-1, \quad u^2=-s^2z^2. \end{array}$$

6 Splitting idempotent

Definition 6.1 (Eilenberg 1948, Cartan & Eilenberg 1956). Let R be a commutative ring. An exact sequence of homomorphisms of R-modules, $\mathrm{im}\, s=\ker\, r$,

$$0 \longrightarrow X' \xrightarrow{s} X \xrightarrow{r} X'' \longrightarrow 0 \tag{2}$$

splits if $\mathrm{im}\, s=\ker\, r=X'$ is a direct summand of X.

Theorem 6.2 (Cartan & Eilenberg 1956, Scheja et al. 1980). The following statements are equivalent

i) The sequence (2) splits.
ii) $\exists g''\in\mathrm{Hom}\,(X'',X)$ with $r\circ g''=\mathrm{id}_{X''}$.
iii) $\exists g\in\mathrm{Hom}\,(X,X')$ with $g\circ s=\mathrm{id}_{X'}$.

Definition 6.3 (Pierce 1982; Hahn 1994). Let $\Delta^\xi\in\mathrm{alg}(\mathcal{C}\ell,\mathcal{C}\ell\otimes\mathcal{C}\ell)$ split the exact sequence of $\mathbb{R}$-algebra homomorphisms

$$0 \longrightarrow \ker\wedge^\eta \xrightarrow{\Delta^\xi} \mathcal{C}\ell\otimes\mathcal{C}\ell \xrightarrow{\wedge^\eta} \mathcal{C}\ell \longrightarrow 0.$$

Thus $\wedge^\eta\circ\Delta^\xi=\mathrm{id}_{\mathcal{C}\ell}$. In this case the element of the crossed $\mathbb{R}$-algebra, viz. $\Delta^\xi 1=\xi^\wedge\in\mathcal{C}\ell\otimes\mathcal{C}\ell$, is said to be a *splitting idempotent* (a cleft of $\mathcal{C}\ell\otimes\mathcal{C}\ell$), $\xi^\wedge=\Delta^\xi 1=\Delta^\xi(1\cdot 1)=(\xi^\wedge)^2$.

If $\wedge^\eta \in \text{alg}(\mathcal{C}\ell \otimes \mathcal{C}\ell, \mathcal{C}\ell)$ then $(\mathcal{C}\ell \otimes \mathcal{C}\ell) \cdot (1 \otimes 1 - \xi^\wedge) \subset \ker \wedge^\eta$.

A crossing defined in Fig. 8 gives a cogebra map $\wedge^\eta \in \text{cog}(\mathcal{C}\ell \otimes \mathcal{C}\ell, \mathcal{C}\ell)$ and an algebra homomorphism $\Delta^\xi \in \text{alg}(\mathcal{C}\ell, \mathcal{C}\ell \otimes \mathcal{C}\ell)$ [Oziewicz 1997, 2001, Cruz & Oziewicz 2000]. However an algebra homomorphism Δ^ξ in general does not need to split.

Definition 6.4. A Clifford convolution $\mathcal{C}\ell(\eta = -\eta^T, \xi = -\xi^T)$ is said to be a Graßmann convolution.

Theorem 6.5. If $\mathcal{C}\ell(\eta, \xi)$ is a Graßmann Hopf gebra, $\dim_{\mathbb{R}} \mathcal{C}\ell = 4$, then Δ^ξ splits, $\wedge^\eta \circ \Delta^\xi = \text{id}_{\mathcal{C}\ell}$, iff

$$\begin{aligned} &\text{either} \quad \text{tr}(\xi \circ \eta) = \dim_{\mathbb{R}} V^\wedge, \\ &\text{or} \quad \eta = 0 \text{ or } \xi = 0, \quad \text{and thus} \quad \det(\text{id} - \xi \circ \eta) = 1. \end{aligned}$$

7 Crossing

A crossing for an antipodal convolution is defined on Fig. 8. The crossing σ

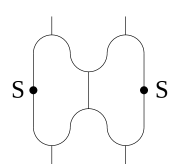

Figure 8: Definition of the crossing σ for antipodal convolution

Fig. 8 is equivalent that there is a cogebra map $\wedge^\eta \in \text{cog}(\mathcal{C}\ell \otimes_\sigma \mathcal{C}\ell, \mathcal{C}\ell)$ and an algebra homomorphism $\Delta^\xi \in \text{alg}(\mathcal{C}\ell, \mathcal{C}\ell \otimes_\sigma \mathcal{C}\ell)$ [Oziewicz 1997, 2001, Cruz & Oziewicz 2000].

A crossing for $\mathcal{C}\ell(0, 0)$, and thus for $\text{tr}(\mu S) = 0$, is the involutive graded switch [Sweedler 1969, Ch.XII],

$$s(x \otimes y) \equiv (-1)^{(\text{grade}\, x)(\text{grade}\, y)} y \otimes x, \quad s^2 = \text{id}_{\mathcal{C}\ell \otimes \mathcal{C}\ell}.$$

In the sequel $\mu \equiv \det(\text{id} - \xi \circ \eta) \neq 0$. The degree of the minimal polynomial of the crossing $\sigma \in \text{End}\,\mathbb{R}(\mathcal{C}\ell \otimes \mathcal{C}\ell)$ we denote by: $\text{degree}(\sigma) \in \mathbb{N}$.

Theorem 7.1 (Oziewicz 1997, p. 1271). Let $\dim_{\mathbb{R}} V = 1$. Then

$$\det \sigma = \left(1 - \frac{2}{\mu}\right)^2, \qquad \operatorname{tr} \sigma = 1 - \mu,$$

$$\begin{array}{rlll} (i) & \text{degree}(\sigma) = 2 & \text{iff} & \mu = 1,\ 2 \text{ or } 4. \\ (ii) & \text{degree}(\sigma) = 3 & \text{iff} & \mu^3 - 4\mu^2 + \mu - 2 = 0. \\ (iii) & \text{degree}(\sigma) = 4 & \text{otherwise.} & \end{array}$$

Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Z}[x]$ be the ring of polynomials in x with coefficients in $\mathbb{Z}$, and $p(x) \in \mathbb{Z}[x]$ be the following polynomial,

$$\begin{aligned} p(x) = \quad & 2^3 - 3^2(17)x + (3)(11)x^2 + (3)(107)x^3 + (2)(97)x^4 \\ & -2^2(3)(17)x^5 - (5)(41)x^6 - (109)x^7 - (167)x^8 \\ & +2^4x^9 + 2^2(3)(7)x^{10} + (37)x^{11} - (2)(3)x^{12} \end{aligned}$$

Theorem 7.2 (Degree of minimal polynomial). Let $\mathcal{C}\ell(\eta, \xi)$ be a Graßmann Hopf gebra, *i.e.* an antipodal Graßmann convolution (Definitions 2.3 & 6.4) with $\operatorname{tr}(\mu S) = 4(\sqrt{\mu} - 1)$ and $\dim_{\mathbb{R}} V = 2$. The minimal polynomial of the crossing is of order 30 in $tr(\eta \circ \xi)$. If $tr(\eta \circ \xi) = 2$ then the minimal polynomial of σ vanishes, as also no antipode exists in this case. Moreover

(i) If $0 \neq (\sqrt{\mu} - 1)$ is not a root of a principal ideal $(p(x)) < \mathbb{Z}[x]$, $p(\sqrt{\mu} - 1) \neq 0$, then the minimal polynomial of the crossing is of degree 3.

(ii) If $p(\sqrt{\mu} - 1) = 0$, then the the minimal polynomial of the crossing is of degree ≤ 2.

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References

[1] Rafał Abłamowicz, CLIFFORD, an Maple V package and BIGEBRA, available at `http://math.tntech.edu/rafal/`

[2] Nicolas Bourbaki, *Elements of Mathematics, Algebra I*, Chapter III, Springer Verlag, Berlin 1989

[3] Henry Cartan & Samuel Eilenberg, *Homological Algebra*, Princeton University Press 1956

[4] José de Jesús de Cruz Guzman and Zbigniew Oziewicz, *Unital and antipodal biconvolution and Hopf gebra*, (2000) this volume

[5] Micho Đurđevich, *Braided Clifford algebras as braided quantum groups*, Advances in Applied Clifford Algebras **4** (2) (1994) 145–156

[6] Micho Đurđevich, *Braided Clifford algebras as quantum deformations*, International Journal of Theoretical Physics **40** (1) (2001)

[7] Samuel Eilenberg, *Extensions of general algebra*, Annales Soc. Polon. Math. (Rocznik Polskiego Towarzystwa Matematycznego) **21** (1) (1948) 125–134

[8] Bertfried Fauser, *On an easy transition from operator dynamics to generating functionals by Clifford algebras*, Journal of Mathematical Physics **39** (1998) 4928–4947

[9] Bertfried Fauser and Rafał Abłamowicz, *On the decomposition of Clifford algebras of arbitrary bilinear form*, in: "Clifford algebras and their applications in mathematical physics", Rafał Abłamowicz and Bertfried Fauser, Eds., Birkhäuser, Boston 2000 p. 341–366

[10] Bertfried Fauser, *On the Hopf algebraic origin of Wick normal-ordering*, 2000a, J. Phys. A.: Math. and Gen. accepted (hep-th/0007032)

[11] Bertfried Fauser, *Quantum Clifford Hopf Gebras for Quantum Field Theory*, talk at "Quantum Group Symmposium" at GROUP23, Dubna, 2000b, (hep-th/0011026)

[12] Alexander J. Hahn, *Quadratic Algebras, Clifford Algebras, and Arithmetic Witt Groups*, Springer Verlag, Berlin 1994, Chap. 2

[13] Gregory Maxwell Kelly & Miguel L. Laplaza, *Coherence for compact closed categories*, Journal of Pure and Applied Algebra **19** (1980) 193–213

[14] Volodimir Lyubashenko, *Tangles and Hopf algebras in braided categories*, Journal of Pure and Applied Algebra **98** (1995) 245–278

[15] Florian Nill, *Axioms for weak bialgebras*, q-alg/9805104

[16] Florian Nill, K. Szlachányi & H.-W. Wiesbrock, *Weak Hopf algebras and reducible Jones inclusion of depth 2. I: From crossed products to Jones towers*, q-alg/9806130

[17] Zbigniew Oziewicz, Eugen Paal and Jerzy Różański, *Derivations in braided geometry*, Acta Physica Polonica **26** (7) (1995) 1253–1273

[18] Zbigniew Oziewicz, *Clifford Hopf gebra and bi-universal Hopf gebra*, Czechoslovak Journal of Physics **47** (12) (1997) 1267–1274

[19] Zbigniew Oziewicz, *Guest Editor's Note: Clifford Algebras and Their Applications*, International Journal of Theoretical Physics **40** (1) (2001) 1–11

[20] Richard S. Pierce, *Associative Algebras*, Graduate Texts in Mathematics # 88, Springer-Verlag, New York 1982

[21] Gian-Carlo Rota & Joel A. Stein, *Plethystic Hopf algebras*, Proceedings of Natl. Academy of Sciences USA **91** (1994) 13057–13061

[22] Jerzy Różański, *Braided antisymmetrizer as bialgebra homomorphism*, Reports in Mathematical Physics **38** (2) (1996) 273–277

[23] Günter Scheja, Uwe Storch, *Lehrbuch der Algebra*, Teil 1, B.G. Teubner, Stuttgart, 1980

[24] Moss E. Sweedler, *Hopf Algebras*, Benjamin, New York 1969

[25] Stanisław Lech Woronowicz, *Differential calculus on compact matrix pseudogroups (quantum groups)*, Communications in Mathematical Physics **122** (1989) 125–170

[26] Stanisław Zakrzewski, *Quantum and classical pseudogroups. Part I. Union pseudogroups and their quantization*, Communications in Mathematical Physics **134** (1990) 347–370